ABSTRACTING THE ANGLE CONCEPT

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Twenty four Year 2 children were presented with realistic models of either cricket or tiling and asked firstly, to indicate which of a set of 10 abstract angle models could represent the path of the ball or the corners of the tiles and secondly, to draw the paths or corners. Responses were analysed to indicate obstacles to abstracting the angle concept. In the cricket context, children needed to conceptualise the path of an object as a straight line segment and then to link the segments to the sequence of actions. In the corners context, children needed to abstract the two sides and ignore the shape at the point; ideas of size were apparently abstracted concurrently. Similar investigations in further contexts promise to uncover other difficulties which children face in abstracting the angle concept.

The research to be described below arises from a model of conceptual development in mathematics developed by Paul White and myself (White & Mitchelmore 1992). This model is based on the assumption that children develop mathematical concepts by abstracting the common features of various situations and learning to ignore the specifics (Skemp, 1971). But concept formation is not a once and for all process; as more and more dissimilar situations are seen to contain the same common elements, the concept becomes more and more general (Mitchelmore 1992). A common path is for the same concept to develop separately in different contexts which the learner does not link together because of superficial differences; generalisation occurs when the superficial differences are seen to be less important that the deep structure.

The angle concept is a case in point. Elsewhere, I have described 14 categories of angle contexts which would seem *a priori* to be superficially different (Mitchelmore 1993). In an initial investigation into how children abstract and generalise the angle concept from these contexts, six were selected and presented to a sample of Year 2 students. This paper reports on their understanding of angle in two of these contexts: cricket and tiling.

METHOD

The cricket context was presented in a model in which a ball was rolled along a groove to a "batter" - a block of wood, faced in plastic foam, which could be rotated to reflect the ball to roll in various directions. A second block of wood representing a fielder was used to provide a target. Children were asked to adjust the batter to hit the ball to the fielder, once on each side of the field and once back to the bowler.

The tiling context employed plastic rhombuses of side 5 cm and angles 60°, 75° and 90°. After making a flooring pattern using nine 75° tiles, children were asked why neither of the other two tiles would fit the pattern, the interviewer steering the discussion towards the corners of the tiles if necessary. A deformable model of a 5 cm rhombus made from meccano was then presented and children were asked to use it to show each tile; they were also asked what the corners of "the funniest tile they could imagine" would look like.

After these different initial introductions, intended to test children's concrete understanding of each context, the interview proceeded in the same way in both contexts. Firstly, in an attempt to assess the extent of each context presented, the interviewer asked children to name examples of "anything else where something is hit like this" (demonstrating the path of the cricket ball) or "corners like these" (holding up the acute-angled corners of the three tiles). Secondly, children were shown the ten abstract models illustrated in Figure 1. The interviewer asked children which of the models could be used to show paths or corners, and to select the one they thought was the best model for doing this. Children were asked to demonstrate how each selected model showed a cover drive or a 75° corner, and how their best choice showed the other hits or tiles. Thirdly, children were asked to draw the various hits or corners.

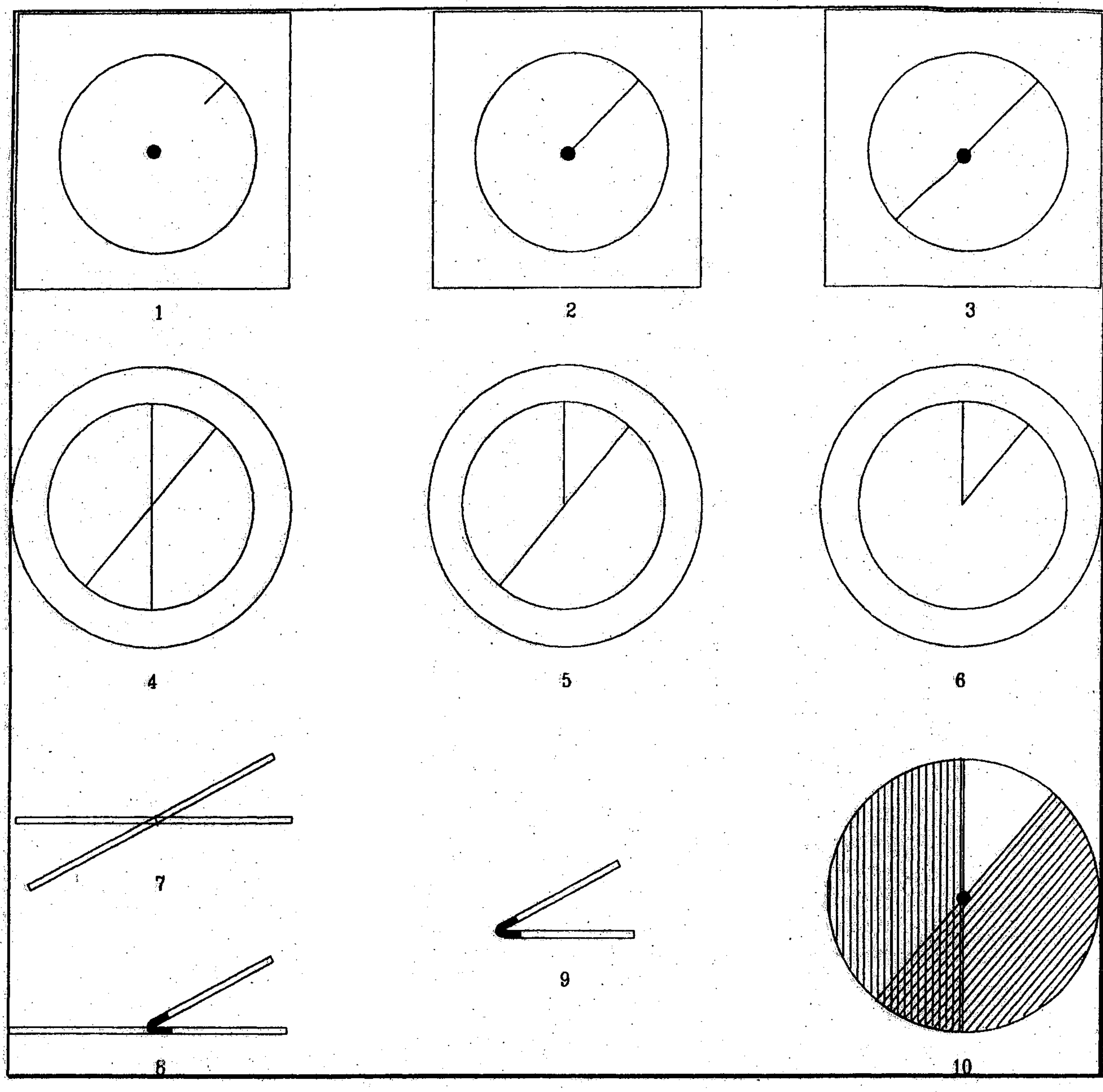

Figure 1 Abstract angle models. Models 1-3 were each made from a transparent plastic circle fastened at its centre to a cardboard square and rotatable about that point; the markings were all on the circles. Models 4-6 were each made from two concentric plastic circles slotted together so that they can rotate relative to each other; one line was marked on each circle. Models 7-9 were made of straws joined together either with thread (7) or a pipe cleaner (8,9). Model 10 was made from two plastic circles, each with one semicircle shaded black, fastened at and rotatable about their common centre.

For each context, twelve children were interviewed; their average age was 7.4 years. Students came from two Catholic schools in northern North Shore suburbs of Sydney.

RESULTS

Cricket

All 12 children were keen to try out what was certainly an unfamiliar cricket model, and they performed fairly well. Five set the batter correctly on the first or second attempt on both sides, whereas only 3 needed more than two attempts on both sides; 7 hit the ball back to the bowler at the first attempt.

All children found other examples of "things being hit", but only other ball games such as soccer and football.

Choice of abstract model suggested, on the other hand, that children has only just begun to abstract the angle implicit in the movement of the ball. Only 5 children modelled the rebound correctly with at least one model, but one of these selected only one model and a second did not select a correct model as the best. The 4 children whose best model correctly modelled the path chose models 5, 6 and 10 (twice). It is notable that, of these 4 children, only the one who had selected only one model had had any difficulty setting the batter in the introduction.

It is instructive to examine the various non-standard ways in which children modelled the path of the ball, as shown by how they used or rejected the models offered. Three methods were employed by more than one student:

- Four children at least once modelled a ball or a rolling action, or rejected a model because it could not roll or fell down. The path of the ball was shown by action on the cricket model rather than in the abstract model.
- Three children at least once used an abstract model to represent only one part of the path (bowler-batter or batter-fielder). Two children set models 7 or 9 to a straight line, one used the line in model 3, and one set models 6 and 9 to an angle of about 120°.
- Three children indicated on models 5 or 8 that the ball started at one end of the longer line, travelled to the batter at the other end, and then back to the centre and out to the fielder along the shorter line. Three of these children also rejected models 4 or 7 because half of one line would have no function; as one put it, *there aren't two fielders*.

Four other ingenious solutions were found to the problem of representing the path of the ball using the two lines in models 4-10, but none showed a single change of direction at the intersection of the two lines.

Children's drawings illustrated further difficulties in abstracting angles from cricket-like contexts. Only 6 of the 12 children drew the standard two line segments, 2 using broken lines. Of the others, 2 students drew only one position of the ball; both showed its path on their pictures with their fingers but could only draw its motion by depicting "air" behind it. Three students drew several different positions of the ball to indicate its movement, and one finished with an arrow pointing from the batter to the fielder. Only one student made a purely abstract drawing which dispensed with the players and the field.

Drawing appeared unrelated to other aspects of abstraction. Of the 6 children who made a standard drawing of the path of the ball, only 2 had chosen abstract models appropriately and only one had shown no difficulty in setting the batter initially.

Tiling

All 12 children seemed to understand completely the effect of the size of the corners of a tile on its fitting into a tiling pattern, even though all of them initially had some difficulty making a pattern with the 75° tiles.

All saw immediately that the proffered square would not fit into the space left by a damaged 75° tile, and all gave some valid reason. The most common global explanation was *it's a square not a diamond* or simply *it's the wrong shape*; but a few referred to the sides as *straight not slant*[ing], *too up* [sloping], or *over a bit*. In the only use of the word "angle" in the whole investigation, one student explained that *the diagonal line is at an angle*.

It was equally obvious to the children that a 60° tile would also not fit. Some explanations compared the length of the major axis of the two rhombi (*it's longer, bigger*) but most focussed on their minor axes (*skinnier, squashed-in, thinner, too much of a diamond, they don't go out the same*). A few comments (*too low, too high, slantier, too much pointed out*) referred to the sides.

All children also found words to compare the corners of the three tiles (*sharper, pointier, smaller, wider, spiky*). Some tested their sharpness with their fingers (one student looking intensely at the depth of the indentations on her skin), but this test did not discriminate as well as a visual test.

All children were also able to adjust the flexible rhombus to the shapes of the three given tiles and to demonstrate the sharpest tile the model would make; however, one objected that the ends were rounded and another pointed out that the rhombus formed by the inside edges of the sides would vanish when it was squashed up. Most could also imagine the sharpest possible tile, describing it variously as *really skinny, one bar of metal* or *like a bumpy stick*. One student objected that it would not be pointy any more because it would be a straight line.

Other examples of corners were easily found, including frequent mention of abstract shapes such as the triangle. However, several examples (*foil, paper, broken glass, sharp metal bits of doors*) suggested that children's idea of sharpness related to the fineness with which a corner was machined rather than to the inclination of its component edges - a point which reoccurs below.Of the 12 students, 8 indicated an appropriate modelling of tile corners using their best model (models 4, 6, 9 or 10 were each chosen twice). However, 4 of these 8 students also omitted at least one appropriate model (other than models 1-3, where the angle is not at all obvious) or used at least one model inappropriately.

There were two main categories of non-standard modelling, both very revealing.

- In model 9, 3 children were influenced by the shape of the pipe cleaner when the model was bent to show an acute angle. One student commented *it get's blunter each time you bend it* [i.e. the more you bend it], compared it to the sharp ends of the straw, and continued *it's only a tiny, winsy bit like it ... it's hard to get it up so high* [i.e. right into the corner]. Two of these children set model 9 inside the corner of the tile without aligning the straws with the edges of the tile.
- Five children set, on at least one model, the vertex of a tile corner at the intersection of two lines but only one edge against a line. Four of these children placed the corner at the intersection of a radius and the circumference in models 2, 3 or 10, and 2 children set it at the centre of models 6 or 10.

Children's drawings indicated other difficulties in abstracting the angle concept from corners. Nine draw conventional angles in the "hat" orientation, but 3 felt the need to complete the triangle. Only 3 children draw all three corners within 15° of their actual size. Two children drew all corners within a range of 10°; both of these had modelled only one edge of each tile corner and had not differentiated different tiles. All but one of the remaining 7 children correctly represented the order of size of the three corners; the median angles drawn by inaccurate children were 34°, 47° and 63° compared to the actual 60°, 75°, 90°.

It may be significant that the 3 accurate drawers all drew the right angled corner with one line vertical and that they were the only ones to do so.

Drawings also varied in the length of the arms (from 1 mm to 58 mm) but there was no obvious correlate of students' preferred lengths. Seven children draw angles which varied markedly in arm length, but in only 3 of these cases were the arm lengths in the same order as the angle sizes.

DISCUSSION

Children's attempts to represent the cricket and tiling contexts using the abstract angle models and in drawings revealed a great deal about their difficulties in abstracting the angle concept - both in seeing the common features and in ignoring the distracting features.

In the cricket context, the initial difficulty seems to be in representing the path of a ball by a straight line. The students in the present sample were well on their way. For although only 4 students correctly modelled the path, all but one of the remaining 8 tried to use lines on the model at least once to represent at least part of the path.

Also, whereas only 7 children drew line segments to represent the path of the ball, only 2 knew of no way to depict the movement.

But that is by no means the end of the story. The learner must recognise that the path only contains two line segments, that they must be joined at their ends at a point which represents the point of deflection, and that extensions of the line segments can be ignored. It is a long way to the level of understanding of the student who remarked that model 4 could represent the path of a cricket ball in 8 different ways. It may be noted that drawing does not focus attention on these abstractions, since children can represent the bowler, batter and fielder; then, as soon as they have learnt the convention of representing a moving object by a line, they automatically produce the correct configuration. It should therefore not be surprising that children who represent an angled path in a standard manner cannot necessarily represent it using an abstract physical model.

The tiling context revealed quite different problems. Here, the main difficulty for children seems to be in abstracting the two sides of the corner as the critical features, ignoring the exact configuration at their intersection. This difficulty might have been heightened in the present investigation by the finely cut corners of the tiles used; but we note that in another context investigated but not reported here (corners in a road), children had an exactly similar difficulty in ignoring the *curved* part of the road at the vertex of a corner. Children's tendency to focus on the vertex of a corner rather than on its sides confirms Davey and Pegg's (1991) finding that young children tend to think of a corner as a point.

The drawing task presented no difficulty representing the vertex, instead showing that most students were still in the process of abstracting the size attribute of a corner. We note that the physical models did not draw attention to size, since the arms could be easily adjusted to match the angle. Only 4 children spontaneously checked their (inaccurate) freehand drawings in a similar way by placing the corner of the tile on them; two then redrew the corners freehand (accurately) whereas two traced the corners from the tiles. It may be significant that the two who redrew freehand had both chosen appropriate best abstract models whereas the two who traced had not; the one student who represented all corners accurately at the first attempt had also chosen an appropriate best abstract model. The abstraction of angle as consisting of two sides would thus seem closely linked to the abstraction of the size of an angle.

CONCLUSION

The investigation of the cricket and tiling contexts has clearly shown several problems which children have in abstracting the angle concept. The other investigations (into turning, scissors, hills and bends) have shown some similar and some quite different problems, to be reported elsewhere.

The abstract models proved to be a valuable means of investigating children's angle concepts. Apart from models 1-3, which are only suitable for showing turning and then do not emphasize an angle, there was little difference between them; they were chosen with approximately equal frequency and were approximately equally often correct. We anticipate that the models could be useful aids in helping children to abstract the angle concept from various contexts and to link the various contexts together to form a general angle concept. It is our next aim to develop and test this proposition.

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